Trial Higher School Certificate Examination – Extension I Mathematics

Question One (12 marks) **Marks**

a) Find the perpendicular distance from the point (1, 4) to the line $4y = 3x - 2$ **2**

b) Fully factorise $2x^3 - 128$ **2**

c) Differentiate with respect to x

i) $y = \sin 2x$ **1** ii) $y = \log_e \sqrt{\dfrac{2x-1}{3x+2}}$ **2**

d) Find the remainder when the polynomial $P(x) = x^4 - 2x^3 - 3$ is divided by $(x-2)$. **2**

e) Evaluate $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ **3**

Question 2 (12 marks) Start on a new page

a) Find the acute angle between the lines $2x + y = 17$ and $3x - y = 3$ **2**

b) Evaluate $\displaystyle\int_0^3 \frac{y}{\sqrt{y+1}}\,dy$, using the substitution $y = u^2 - 1$ **4**

c) Prove that $\dfrac{\cos 2\theta}{\cos\theta - \sin\theta} = \cos\theta + \sin\theta$ **3**

d) Find all solutions to: $\dfrac{1}{x(2-x)} \leq 0$ **3**

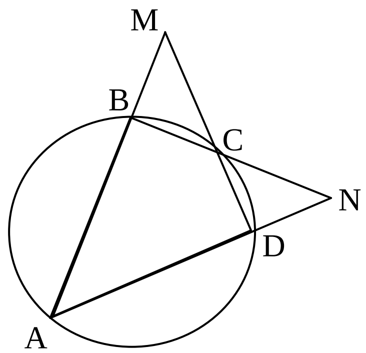
Question 3 (12 marks) **Marks**

a) P (−7, 3), Q (9, 15) and B (14, 0) are three points and A divides the interval PQ in the ratio 3:1.
Prove that PQ is perpendicular to AB. **3**

b) If $f(x) = 2\sin^{-1}(3x)$ find the domain and range of $f(x)$ **2**

c) Find the value of the constant term in the expansion of $\left(3x + \frac{2}{\sqrt{x}}\right)^6$ **2**

d)



In the figure *ABM*, *DCM*, *BCN* and *ADN* are straight lines and angle *AMD* = angle *BNA*.

i) Copy the diagram onto the answer sheet and prove that angle *ABC* = angle *ADC*. **3**

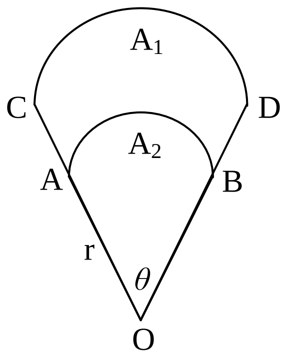
ii) Hence, or otherwise prove that *AC* is a diameter. **2**

Question 4 (12 marks)

a) Given $f(x) = \log_e(e^x + 1)$

i) Find the area bounded between $y = f(x)$, the x axis, $x = 1$ and $x = 5$ by the use of five ordinate values of Simpson's rule. Leave your answer correct to 2 decimal places. **3**

ii) By considering the second derivative of $f(x)$, state whether the application of the trapezoidal rule would over or under estimate the actual area described in part (i) above. Justify your answer. **3** b)



The diagram shows sector *OCD* which is formed by a piece of wire and a concentric arc *AB*, which is formed by a second piece of wire. The length of *OA* is *r*.

i) Show that, if the area A_1 of region *ABDC*, is three times the area A_2 of region *OAB*, then $AC = r$. **2**

ii) If the total length of the two pieces of wire forming the figure is 24 cm, show that $\theta = \dfrac{24 - 4r}{3r}$. **2**

iii) Hence find the value of *r* so that the area of the sector *OCD* is a maximum. **2**

Question 5 (12 marks)

a) Solve for $0 \le \theta \le \pi$, $\cos\theta + 3\sin\dfrac{\theta}{2} - 2 = 0$ **3**

b) i) Sketch the curve $y = 4\tan^{-1}(x)$ clearly showing its range. **2**

ii) Find the volume of the solid formed when the area bounded by the curve $y = 4\tan^{-1} x$, the *y* axis, the line $y = \pi$ is rotated one revolution about the y – axis. **3**

c) i) The acceleration in ms^{-2} of an object is given by $\ddot{x} = 2x^3 + 4x$.
If the object is initially 2 *m* to the right of the origin travelling with velocity $8ms^{-1}$, find an expression for v^2 (the square of its velocity) in terms of *x*. **2**

ii) What is the minimum speed of the object?
Give a reason for your answer. **2**

Question 6 (12 marks) **Marks**

a) Steven borrows \$50 000 to pay for a new car. He plans to repay the loan by making 60 equal monthly instalments. Interest is charged at the rate of 0.6% per month on the balance owing.

i) Show that immediately after making two monthly instalments of \$M, the balance owing is given by \$(50601.80 – 2.006M) **2**

ii) Calculate the value of each monthly instalment. **2**

b) Prove by Mathematical Induction that:

$2\,(1!) + 5\,(2!) + 10\,(3!) + \ldots + (n^2 + 1)\,n! = n\,(n + 1)!$ for positive integers $n \geq 1$. **4**

c) If $y = \dfrac{\log_e x}{x}$

i) Show that $\dfrac{dy}{dx} = \dfrac{1 - \log_e x}{x^2}$ **1**

ii) and hence or otherwise show that

$$\int_e^{e^2} \frac{1 - \log_e x}{x \log_e x}\,dx = \log_e 2 - 1$$ **3**

Question 7 (12 marks)

a) The deck of a ship was 1.4 m below the level of a wharf at low tide and 0.4 m above wharf level at high tide. Low tide was at 8:24 am and high tide at 2:40 pm. If the tide's motion is simple harmonic, find the first time after 8:24 am that the deck was level with the wharf. **5**

b) i) Show that $(1 + x)^m \left(1 - \dfrac{1}{x}\right)^m = \left(x - \dfrac{1}{x}\right)^m$ **2**

ii) By considering the term(s) independent of *x* in the expansion of part b (i), justify the result:

$$\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \ldots. + \binom{2002}{2002}^2 = -1\binom{2002}{1001}$$ **5**

END OF PAPER

Trial Higher School Certificate Examination – Extension I Mathematics Solutions

Question 1

a)

$$D_{\perp} = \left|\frac{3(1) - 4(4) - 2}{\sqrt{9+16}}\right| \qquad 3x - 2 - 4y = 0$$

$$= \left|\frac{-15}{\sqrt{25}}\right|$$

$$= \frac{15}{5}$$

$$= 3$$

b)

$$2\left(x^3 - 64\right)$$

$$= 2\,(x-4)\left(x^2 + 4x + 16\right)$$

c) i) $y' = 2\cos 2x$

$$ii)\, y = \log_e \left(\frac{2x-1}{3x+2}\right)^{½}$$

$$y = \tfrac{1}{2}\left[\log_e\,(2x-1) - \log_e\,(3x+2)\right]$$

$$y' = \tfrac{1}{2}\left[\frac{2}{2x-1} - \frac{3}{3x-2}\right]$$

$$= \frac{7}{2(2x-1)(3x+2)}$$

d) $P(2) = 2^4 - 2(2)^3 - 3 = -3$

e)

$$e)\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}}$$

$$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\,(\frac{1}{\sqrt{2}})$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Question 2

a)

$$\tan\theta = \left|\frac{-2-3}{1+(-6)}\right|$$

$2x + y = 17 \Rightarrow \ m_1 = -2$

$3x - y = 3 \Rightarrow \ m_2 = 3$

$$= 1$$

$$\therefore \theta = 45°$$

b)

$$\int_0^3 \frac{y}{\sqrt{y+1}}\,dy$$

$$= \int_1^2 \frac{u^2 - 1}{u}\, 2u \cdot du$$

$$= 2\int_1^2 \left(u^2 - 1\right) du$$

$$= 2\left[\frac{u^3}{3} - u\right]_1^2$$

$$= 2\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)$$

$$= 2\frac{2}{3}$$

$y = 0,\ u = 1$

$y = u^2 - 1 \quad y = 3,\ u = 2$

$$\frac{dy}{du} = 2u$$

$$\therefore dy = 2u\,du$$

c)

$$\frac{\cos 2\theta}{\cos\theta - \sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}$$

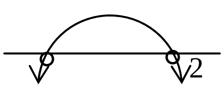
$$= \cos\theta + \sin\theta$$

$$= RHS$$

QED

d)

$$\frac{1}{x(2-x)} \le 0 \quad x \ne 0,\ 2$$



$$x(2-x) \le 0$$

$$x = 0 \ \ x = 2 \quad S = \{x : x < 0,\ x > 2\}$$

Question 3

a) P (−7, 3) Q (9, 15) B (14, 0)

$$\times$$
$$3:1$$

$$A\left(\frac{-7+27}{4}, \frac{3+45}{4}\right) = A(5, 12)$$

$$M(PQ) = \frac{15-3}{9+7} = \frac{3}{4}$$

$$M(AB) = \frac{12-0}{5-14} = \frac{-4}{3}$$

$$M(PQ)\ M(AB) = \frac{3}{4} \times \frac{-4}{3} = -1$$

$$\therefore PQ \perp AB\ (prod\ of\ M = -1)$$

b)

Domain $-\frac{1}{3} \le x \le \frac{1}{3}$

Range $-\pi \le y \le \pi$

c)

$$T_{r+1} = {}^6C_r\ (3x)^{6-r} \left(\frac{2}{\sqrt{x}}\right)^r$$

$$= {}^6C_r\ 3^{6-r}\ 2^r\ x^{6-r}\ x^{-\frac{1}{2}r}$$

$$= {}^6C_r\ 3^{6-r}\ 2^r\ x^{6-1\frac{1}{2}r}$$

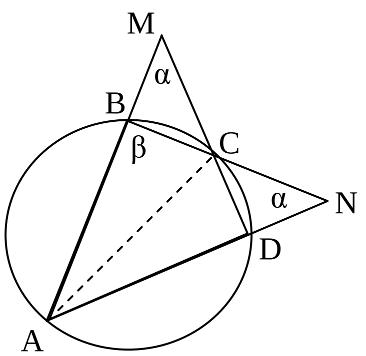
for constant term degree of $x = 0$

$\therefore 6 - 1\frac{1}{2}r = 0$

$r = 4 \quad \therefore {}^6C_4\ 3^2\ 2^4$

$= 2160$

d)



i) Let $\angle AMD = \alpha = \angle ANB$ and $\angle ABN = \beta$

$\angle BCM = \beta - \alpha$ (ext $\angle$ of Δ)

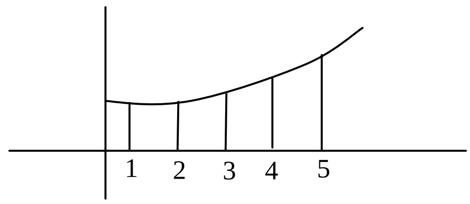
$\angle DCN = \beta - \alpha$ (vert opp)

$\therefore \angle ADC = \beta$ (ext $\angle \Delta$)

ii) $\angle ABC + \angle ADC = 180°$ (opp angle of cyclic quad supple)

$\therefore 2\beta = 180° \quad \therefore \beta = 90° \quad \therefore AC$ is diam.

Question 4



a)

$$A \doteqdot \frac{b-a}{6}\left[f(a) + 4\text{odds} + 2\text{evens} + f(b)\right]$$

$$= \frac{1}{3}\left[\ln(e+1) + 4\ln(e^2+1) + 2\ln(e^3+1) + 4\ln(e^4+1) + \ln(e^5+1)\right]$$

$$= \frac{1}{3}\left[1 + 8.5077 + 6.09717 + 20.02686 + 1.79175\right]$$

$$= 12.4745$$

x	1	2	3	4	5
y	$\ln(e+1)$	$\ln(e^2+1)$	$\ln(e^3+1)$	$\ln(e^4+1)$	$\ln(e^5+1)$

iii)

$$f(x) = \ln(e^x + 1)$$

$$f'(x) = \frac{e^x}{e^x + 1}$$

$$f''(x) = \frac{e^x}{(e^x+1)^2} > 0 \text{ for all } x$$

$\therefore$ $f(x)$ is concave up

$\therefore$ trapezium rule will over estimate area.

b) i) $A_1 = 3A_2$

Now $A_2 = \frac{1}{2} r^2\theta$

$\therefore A_1 = \frac{3}{2} r^2 \theta$

but $A_1 = A_r - A_2$ where A_r is area of entire region

$$= \frac{1}{2}\left(OC^2\,\theta - r^2\,\theta\right)$$

$$\frac{3}{2} r^2\theta = \frac{1}{2}\theta\left(OC^2 - r^2\right)$$

$$3r^2 = OC^2 - r^2$$

$$4r^2 = OC^2$$

$$\therefore OC = 2r$$

but $OC = AC + AO$

$$2r - r = AC$$

$$\therefore AC = r$$

ii)

$$l_1 = 2r\theta$$

$$l_2 = r\theta$$

$$\therefore tot\ l = 2r + 2r + 2r\theta + r\theta$$

$$24 = 4r + 3r\theta$$

$$\therefore \frac{24 - 4r}{3r} = \theta$$

Question 4 continued.

iii)

$$A = \frac{1}{2} r^2 \ \theta$$

$$= \frac{1}{2} r^2 \left(\frac{24 - 4r}{3r} \right)$$

$$= \frac{24}{6} r - \frac{2r^2}{3} \qquad\qquad A'' = \frac{-4}{3} \quad \therefore \text{ max area, concave down}$$

$$A' = 4 - \frac{4r}{3}$$

$$\therefore \frac{4r}{3} = 4$$

$r = 3$ gives maximum area

Question 5

a)

$$\cos\theta + 3\sin\frac{\theta}{2} - 2 = 0$$

$$1 - 2\sin^2\frac{\theta}{2} + 3\sin\frac{\theta}{2} - 2 = 0$$

$$2\sin^2\frac{\theta}{2} - 3\sin\frac{\theta}{2} + 1 = 0$$

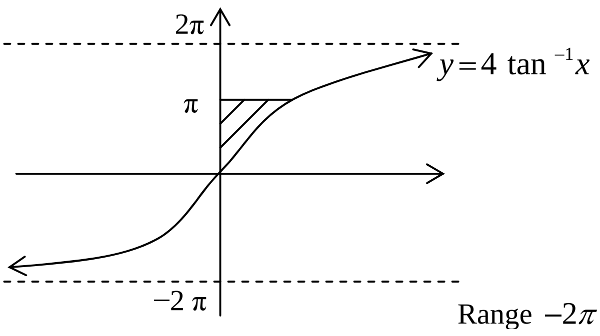
$$\left(2\sin\frac{\theta}{2} - 1\right)\left(\sin\frac{\theta}{2} - 1\right) = 0$$

$$\sin\frac{\theta}{2} = \frac{1}{2}, 1$$

$$\therefore \frac{\theta}{2} = \frac{\pi}{6}, \frac{\pi}{2}$$

$$\therefore \theta = \frac{\pi}{3}, \pi$$

b) i)



Range $-2\pi < y < 2\pi$

Question 5 continued

b) ii)

$$\frac{y}{4} = \tan^{-1} x$$

$$x = \tan \frac{y}{4}$$

$$V = \pi \int_0^{\pi} x^2 \, dy$$

$$= \pi \int_0^{\pi} \tan^2 \frac{y}{4} \, dy$$

$$= \pi \int_0^{\pi} (\sec^2 \frac{y}{4} - 1) dy$$

$$= \pi \left[4 \tan \frac{y}{4} - y \right]_0^{\pi}$$

$$= \pi \left[\left(4 \tan \frac{\pi}{4} - \pi \right) - \left(4 \tan 0 - 0 \right) \right]$$

$$= \pi \left(4 - \pi \right) u^3$$

c) i)

$$\ddot{x} = 2x^3 + 4x$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 2x^3 + 4x$$

$$\frac{1}{2}v^2 = \frac{x^4}{2} + 2x^2 + c$$

$$t = 0, \; x = 2, \; v = 6$$

$$\therefore 18 = 8 + 8 + c$$

$$c = 2$$

$$\therefore v^2 = x^4 + 4x^2 + 4$$

ii)

$$v^2 = \left(x^2 + 2\right)^2$$

$$\therefore v^2 \geq 4, \therefore v \neq 0$$

$\therefore$ object never changes direction.

Always moves to right with increasing speed since initial velocity >0 and acceleration >0 for $x > 0$ $\therefore$ min speed is initial speed.

$\therefore$ min speed is $6 ms^{-1}$.

Question 6

a) i) Amount owing after 1st payment: = \$50000 × 1.006 − M.

Amount owing after 2nd payment = \$50000 × 1.006 − M (1 + 1.006)

Balance = \$50601.80 − 2006 M

ii)

$$30000 \times 1.006^{60} - \frac{M\left(1.006^{60} - 1\right)}{0.006} = 0$$

$$\therefore M = \frac{50000 \times 1.006}{\frac{1.006^{60} - 1}{0.006}}$$

$$\therefore M = \$994.78$$

b)

When $x = 1$, $LHS = 2(1!) = 2$

$RHS = 1(2!) = 2$

$\therefore$ true for $n = 1$

assume true for $n = k$

$ie\ 2(1!) + 5(2!) + ... + \left(k^2 + 1\right)k! = k(k+1)!$

to prove true for $n = k + 1$

$ie\ 2(1!) + 5(2!) + ... + \left(k^2 + 1\right)k! + \left[(k+1)^2 + 1\right](k+1)! = (k+1)(k+2)!$

Now $LHS = 2(1!) + 5(2!) + ... + \left(k^2 + 1\right)k! + \left(k^2 + 2k + 2\right)(k+1)!$

$= k(k+1)! + \left(k^2 + 2k + 2\right)(k+1)!$ by assumption

$= (k+1)!\left\{k + k^2 + 2k + 2\right\}$

$= (k+1)!\left(k^2 + 3k + 2\right)$

$= (k+1)!(k+2)(k+1)$

$= (k+2)!(k+1)$

$= RHS$

$\therefore$ if true for $n = k$, then true for $n = k + 1$ and since true for $n = 1$, then true for all intergers, $n \geq 1$

c) i)

$$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - (1)(\ln x)}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

Question 6 continued

c) ii)

$$\int_e^{e^2} \frac{1 - \ln x}{x \ln x}\, dx = \int_e^{e^2} \frac{\frac{1 - \ln x}{x^2}}{\frac{\ln x}{x}}\, dx$$

$$= \left[\ln\left(\frac{\ln x}{x}\right) \right]_e^{e^2}$$

$$= \ln\left(\frac{\ln e^2}{e^2}\right) - \ln\left(\frac{\ln e}{e}\right)$$

$$= \ln\left(\frac{2}{e^2}\right) - \ln\left(\frac{1}{e}\right)$$

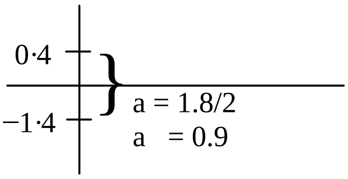
$$= \ln\left(\frac{2}{e}\right)$$

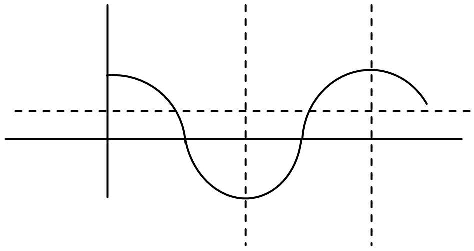
$$= \ln 2 - \ln e$$

$$= \ln 2 - 1$$

Question 7

a)





$$\frac{T}{2} = 6\,hrs\ 16\ \text{min} = 6\frac{4}{15}$$

$$\frac{T}{2} = \frac{2\pi}{2n}$$

$$n = \frac{15\pi}{94}$$

equation of motion

$$x = -a\cos nt$$

$$\therefore 0\cdot 5 = -0.9\cos\frac{15\pi}{94}t$$

$$t = \frac{94}{15\pi}\left[2n\pi \pm \cos^{-1}\left(-\frac{5}{9}\right)\right]$$

$$t = 4.08298$$

= 4 hr18 min.

$\therefore$ First time after low tide deck is level with wharf is 12:42 pm.

b) i)

$$(1+x)^m\left(1-\tfrac{1}{x}\right)^m = \left[(1+x)\left(1-\tfrac{1}{x}\right)\right]^m$$

$$= \left[1-\tfrac{1}{x}+x-1\right]^m$$

$$= \left(x-\tfrac{1}{x}\right)^m$$

ii) Letting $m = 2002$

$$LHS = (1+x)^{2002}\left(1-\tfrac{1}{x}\right)^{2002}$$

$$= \left[\binom{2002}{0}+\binom{2002}{1}x+\ldots\ldots\binom{2002}{r}x^r+\binom{2002}{2002}x^{2002}\right]$$

$$\times\left[\binom{2002}{0}-\binom{2002}{1}\frac{1}{x}+\ldots\ldots+(-1)^r\binom{2002}{r}\frac{1}{x^r}+\ldots..+\binom{2002}{2002}\frac{1}{x^{2002}}\right]$$

Co eff of x° in LHS is

$$\binom{2002}{0}\times\binom{2002}{0}+\binom{2002}{1}\times-\binom{2002}{1}+\ldots.+(-1)^r\binom{2002}{r}^2+\ldots.+\binom{2002}{2002}^2$$

ie. $\binom{2002}{0}^2-\binom{2002}{1}^2+\binom{2002}{2}^2-\ldots.+(-1)^r\binom{2002}{r}^2+\ldots.+\binom{2002}{2002}^2$

Question 7 continued

b) ii) $RHS = \left(x - \frac{1}{x}\right)^{2002}$

Gen term is $\binom{2002}{r} x^{2002-r} \left(-\frac{1}{x}\right)^r$

$= (-1)^r \binom{2002}{r} x^{2002-2r}$

$\therefore$ co eff of x° occurs when $2002 - 2r = 0$

ie. $r = 1001$

$\therefore$ co eff is $(-1)^{1001} \binom{2002}{1001} = -1 \binom{2002}{1001}$

$$\therefore \quad \binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \ldots + \binom{2002}{2002}^2 = -1 \binom{2002}{1001}$$

THE END